Supplementary Material

A Probabilistic Framework for Color-Based Point Set Registration

Martin Danelljan, Giulia Meneghetti, Fahad Shahbaz Khan, Michael Felsberg
Computer Vision Laboratory, Department of Electrical Engineering, Linköping University, Sweden
{martin.danelljan, giulia.meneghetti, fahad.khan, michael.felsberg}@liu.se

In this supplementary material of [1] we provide additional derivations. In section 1 we derive the expression of the latent posteriors, given in eq. (16) in [1]. Section 2 gives a derivation of the loss function stated in eq. (18) in [1]. In section 3 we derive the update of the feature component weights, given in eq. (19) in [1]. Finally, additional analysis of parameter settings is given in section 4.

1. Derivation of the Latent Posteriors $\alpha_{ijkl}^{(n)}$

Here, we derive the expression of the latent posteriors $\alpha_{ijkl}^{(n)}$ given in eq. (16) in [1]. By using the factorization (10) in [1], along with the definitions of the individual factors (see (1), (11) and (12) in [1]) we obtain the following expression of the complete-data likelihood of the observation $(\mathbf{x}_{ij}, y_{ij})$ (also given in (15) in [1] for $k \neq 0$),

$$p(\mathbf{x}_{ij}, y_{ij}, C_{ij} = l, Z_{ij} = k|\Theta) = \begin{cases} \pi_k \rho_{kl} B_l(y_{ij}) \mathcal{N}(\phi_i(\mathbf{x}_{ij}); \boldsymbol{\mu}_k, \Sigma_k)\,, & k \neq 0 \\ \dfrac{\pi_0}{L} \mathcal{U}_U(\phi_i(\mathbf{x}_{ij})) \mathcal{U}_\Omega(y)\,, & k = 0. \end{cases} \tag{1}$$

The observed data-likelihood for $(\mathbf{x}_{ij}, y_{ij})$ is obtained by marginalizing over the latent variables (C_{ij}, Z_{ij}),

$$\begin{aligned} p(\mathbf{x}_{ij}, y_{ij}|\Theta) &= \sum_{k=0}^{K} \sum_{l=1}^{L} p(\mathbf{x}_{ij}, y_{ij}, C_{ij} = l, Z_{ij} = k|\Theta) \\ &= \sum_{k=1}^{K} \sum_{l=1}^{L} \pi_k \rho_{kl} B_l(y_{ij}) \mathcal{N}(\phi_i(\mathbf{x}_{ij}); \boldsymbol{\mu}_k, \Sigma_k) + \pi_0 \mathcal{U}_U(\phi_i(\mathbf{x}_{ij})) \mathcal{U}_\Omega(y). \end{aligned} \tag{2}$$

The set $U \subset \mathbb{R}^3$ is selected to contain all observed points. The last term in (2), which corresponds to the uniform component, is therefore a constant λ, given by

$$\lambda := \pi_0 \mathcal{U}_U(\phi_i(\mathbf{x}_{ij})) \mathcal{U}_\Omega(y) = \frac{\pi_0}{m(U) m(\Omega)}. \tag{3}$$

Here, m denotes the reference measure of the probability densities for the respective spaces (*i.e.* the Lebesgue measure in the spatial case). The latent posteriors are given by the conditional probabilities,

$$\alpha_{ijkl}^{(n)} := p(Z_{ij} = k, C_{ij} = l|\mathbf{x}_{ij}, y_{ij}, \Theta^{(n)}) = \frac{p(\mathbf{x}_{ij}, y_{ij}, C_{ij} = l, Z_{ij} = k|\Theta^{(n)})}{p(\mathbf{x}_{ij}, y_{ij}|\Theta^{(n)})}. \tag{4}$$

Here, $\Theta^{(n)}$ denotes the model parameter estimate obtained in EM-iteration n. By using (1), (2) and (3) in (4) we obtain,

$$\alpha_{ijkl}^{(n)} = \frac{\pi_k^{(n)} \rho_{kl}^{(n)} B_l(y_{ij}) \mathcal{N}\big(\phi_i^{(n)}(\mathbf{x}_{ij}); \boldsymbol{\mu}_k^{(n)}, \Sigma_k^{(n)}\big)}{\sum\limits_{q=1}^{K} \sum\limits_{r=1}^{L} \pi_q^{(n)} \rho_{qr}^{(n)} B_r(y_{ij}) \mathcal{N}\big(\phi_i^{(n)}(\mathbf{x}_{ij}); \boldsymbol{\mu}_q^{(n)}, \Sigma_q^{(n)}\big) + \lambda}\,, \quad k \neq 0. \tag{5}$$

These are the latent posteriors $\alpha_{ijkl}^{(n)}$ obtained in EM-iteration n, also stated in eq. (16) in [1].

2. Derivation of the Loss $g(\Theta; \Theta^{(n)})$

In this section, we derive the loss $g(\Theta; \Theta^{(n)})$ (eq. (18) in [1]) that is used in the M-step of the EM procedure. In the M-step, the aim is to maximize the expected complete-data log likelihood $Q(\Theta; \Theta^{(n)})$. For the proposed model, this is given by eq. (17) in [1],

$$Q(\Theta; \Theta^{(n)}) = \mathrm{E}_{\mathcal{Z}|\mathcal{X},\Theta^{(n)}}\left[\log p(\mathcal{X}, \mathcal{Z}|\Theta)\right] = \sum_{ijkl} \alpha_{ijkl}^{(n)} \log p(\mathbf{x}_{ij}, y_{ij}, C_{ij} = l, Z_{ij} = k|\Theta). \tag{6}$$

By using the formula (1) for the complete-data likelihood of each observation, we obtain

$$\begin{aligned} Q(\Theta; \Theta^{(n)}) = \sum_{ijl} \sum_{k=1}^{K} \alpha_{ijkl}^{(n)} \left(\log \pi_k + \log \rho_{kl} + \log B_l(y_{ij}) - \frac{3}{2} \log \pi - \frac{1}{2} \log |\Sigma_k| - \frac{1}{2} \| R_i \mathbf{x}_{ij} + \mathbf{t}_i - \boldsymbol{\mu}_k \|_{\Sigma_k^{-1}}^2 \right) \\ + \sum_{ijl} \alpha_{ij0l}^{(n)} \log \frac{\lambda}{L}. \end{aligned} \tag{7}$$

Here, λ is the constant defined in (3). To simplify the expression (7), we omit constant terms. In our case, the terms $\log B_l(y_{ij})$ and $\frac{3}{2} \log \pi$ do not depend on the model parameters. The last term is also a constant, since π_0 is a fix meta parameter. By omitting these unnecessary terms in (7), we obtain the equivalent loss,

$$g(\Theta; \Theta^{(n)}) = \sum_{ij} \sum_{k=1}^{K} \sum_{l=1}^{L} \alpha_{ijkl}^{(n)} \left(\frac{1}{2} \log |\Sigma_k| + \frac{1}{2} \| R_i \mathbf{x}_{ij} + \mathbf{t}_i - \boldsymbol{\mu}_k \|_{\Sigma_k^{-1}}^2 - \log \pi_k - \log \rho_{kl} \right). \tag{8}$$

This loss is then employed in the M-step of our EM procedure.

3. Derivation of the Optimal Feature Component Weights $\rho_{kl}^{(n)}$

Here, we derive the formula, stated in eq. (20) in [1], for updating the feature component weights ρ_{kl}. In the M-step of our EM-procedure, the feature component weights ρ_{kl} are updated by minimizing the loss (8) (*i.e*. eq. (19) in [1]). Since only the last term in (8) depends on ρ_{kl}, we obtain the equivalent optimization problem,

$$\text{minimize} \quad \varepsilon = -\sum_{ij} \sum_{k=1}^{K} \sum_{l=1}^{L} \alpha_{ijkl}^{(n)} \log \rho_{kl} \tag{9a}$$

$$\text{subject to} \quad \sum_{l=1}^{L} \rho_{kl} = 1\,, \; k = 1, \ldots, K \tag{9b}$$

Here, the constraints in (9b) ensure that the feature component weights sum up to one. By introducing Lagrange multipliers η_k, we obtain

$$\mathcal{L} = -\sum_{ij} \sum_{k=1}^{K} \sum_{l=1}^{L} \alpha_{ijkl}^{(n)} \log \rho_{kl} + \sum_{k=1}^{K} \eta_k \left(\sum_{l=1}^{L} \rho_{kl} - 1 \right). \tag{10}$$

Differentiation with respect to ρ_{kl} gives,

$$\frac{\partial \mathcal{L}}{\partial \rho_{kl}} = -\frac{1}{\rho_{kl}} \sum_{ij} \alpha_{ijkl}^{(n)} + \eta_k. \tag{11}$$

The optimum $\rho_{kl}^{(n)}$ is obtained by setting the partial derivatives (11) to zero,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \rho_{kl}} &= 0 \iff \\ \rho_{kl}^{(n)} &= \frac{1}{\eta_k} \sum_{ij} \alpha_{ijkl}^{(n)} \end{aligned} \tag{12}$$

The Lagrange multipliers η_k are computed by summing both sides of (12) over l and using the constraint (9b),

$$\sum_{l=1}^{L} \rho_{kl}^{(n)} = \sum_{l=1}^{L} \frac{1}{\eta_k} \sum_{ij} \alpha_{ijkl}^{(n)} \iff$$

$$1 = \frac{1}{\eta_k} \sum_{ij} \sum_{l=1}^{L} \alpha_{ijkl}^{(n)} \iff$$

$$\eta_k = \sum_{ij} \sum_{l=1}^{L} \alpha_{ijkl}^{(n)} = \sum_{ij} \alpha_{ijk}^{(n)} \tag{13}$$

In the last equality we have used the definition $\alpha_{ijk}^{(n)} = \sum_{l=1}^{L} \alpha_{ijkl}^{(n)}$ (section 4.2 in [1]). By using (13) in (12), we obtain eq. (20) in [1] as

$$\rho_{kl}^{(n)} = \frac{\sum_{ij} \alpha_{ijkl}^{(n)}}{\sum_{ij} \alpha_{ijk}^{(n)}}, \; k = 1, \ldots, K. \tag{14}$$

4. Parameter variations

Here, we provide further analysis of the parameters used in our approach. We investigate the impact of varying the number of spatial components K (fig. 1) and outlier ratio parameter π_0 (fig. 2). Our results remain stable to parameter perturbations. Additionally, our method achieves a consistent improvement over JRMPS [2] in both accuracy and robustness, independent of parameter settings.

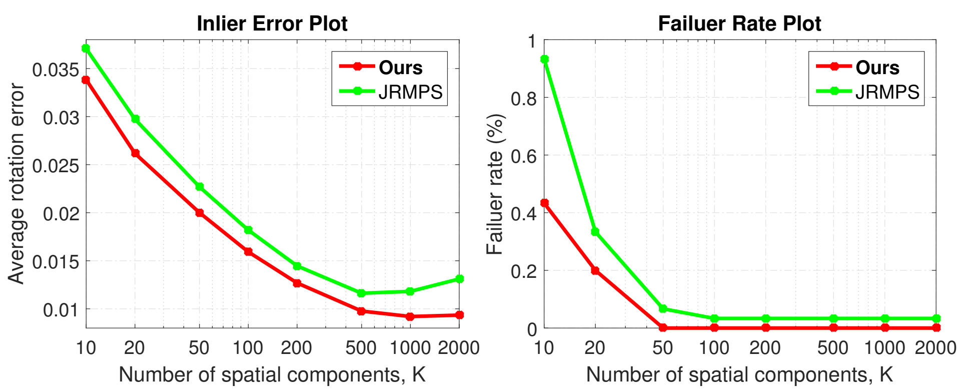

Figure 1. Analysis of the number of spatial components K. We show the average inlier rotation error (left) and failure rate (right) for our color-based method (red) and the baseline JRMPS (green).

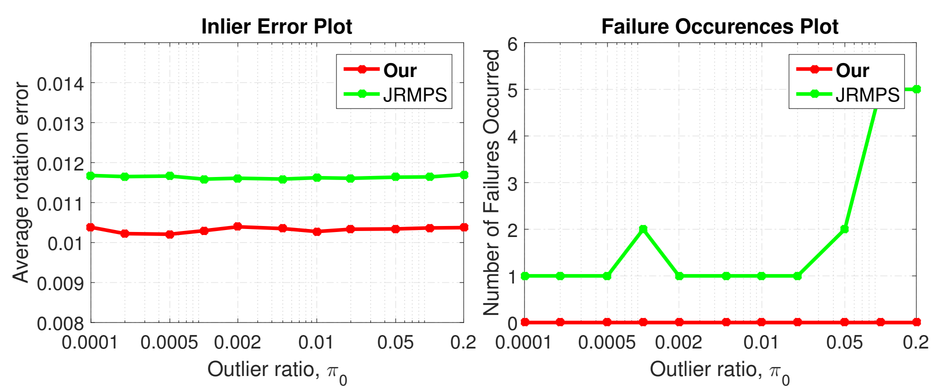

Figure 2. Analysis of the outlier ratio parameter π_0. We show the average inlier rotation error (left) and failure rate (right) for our color-based method (red) and the baseline JRMPS (green).

References

[1] M. Danelljan, G. Meneghetti, F. Shahbaz Khan, and M. Felsberg. A probabilistic framework for color-based point set registration. In *CVPR*, 2016. 1, 2, 3

[2] G. D. Evangelidis, D. Kounades-Bastian, R. Horaud, and E. Z. Psarakis. A generative model for the joint registration of multiple point sets. In *ECCV*, 2014. 3